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Particle-like solution for SO(3) gauge field coupled to strong gravity

I V Krive† and Yu A Sitenko‡

† Kharkov State University, Kharkov, USSR
‡ Institute for Theoretical Physics, Kiev, USSR

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Abstract. The classical equations for the SO(3) gauge field coupled with the strong gravity metric are considered in the f–g theory of Salam *et al*. The asymptotics of the regular static spherically symmetric solutions are obtained.

1. Introduction

Recently there has been considerable interest in classical solutions of field theories with non-Abelian gauge symmetry. The non-linearity of the equations of motion which occurs even in a pure gauge field theory leads to the appearance of non-trivial types of solutions (Yang and Wu 1969, Marciano and Pagels 1975). However, it should be noted that solutions with finite energy for a pure gauge field theory are excluded in space–time of the dimension $d \leqslant 4$ (Deser 1976). For the models involving scalar fields together with gauge fields, stable (due to the conservation of the topological charge (see Tyupkin *et al* 1975, Arafune *et al* 1975)) classical solutions have been found corresponding to extended particles with the properties of the magnetic monopole ('t Hooft 1974, Polyakov 1974) and the dyon (Julia and Zee 1975, Prasad and Sommerfield 1975).

These results have inspired the investigations of these objects in curved space–time. Singular spherically symmetric solutions have been found for the matter fields and metric tensors (Bais and Russel 1975, Cho and Freund 1975) and the existence of regular solutions (i.e. those possessing finite energy) has also been proved (van Nieuwenhuizen *et al* 1976). The regular solutions for the pure Yang–Mills fields interacting with Einstein gravity have been obtained using numerical calculations (Wang 1975, Burlankov and Dutishev 1976). The existence of particle-like solutions in such theories is certainly of interest. However, in the case of conventional gravity the region where the fields differ distinctly from their vacuum values is exceedingly small (with characteristic range of $k_g \equiv \sqrt{G} \sim 10^{-33}$ cm). Therefore it is doubtful that such objects can be relevant to strong interaction physics (where the consideration of massless non-Abelian gauge fields—gluons—makes sense).

It seems more reasonable to investigate regular classical solutions for the non-Abelian gauge fields in the framework of two-tensor (f–g) theory of gravity (Isham *et al* 1971), which describes both strong and gravitational interactions in terms of metric fields. In this case for the particle-like solutions the fields differ from their vacuum values in the range of m^{-1} ($m \sim 1$ GeV is the characteristic hadron mass). Such solutions can be regarded as mesons consisting only of gluons without quarks. In this paper we demonstrate the existence of these solutions for the SO(3) gauge field coupled with strong gravity metric field.

2. Derivation of equations

The Lagrangian density of the system considered is given by

$$\mathcal{L} = -\frac{1}{k_f^2}(-f)^{1/2}R_f - \frac{m^2}{4k_f^2}(-f)^{1/2}(f^{\mu\nu} - g^{\mu\nu})(f^{\rho\sigma} - g^{\rho\sigma})(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma})$$
$$- \frac{1}{k_g^2}(-g)^{1/2}R_g - \frac{1}{4}(-f)^{1/2}f^{\mu\nu}f^{\rho\sigma}\mathcal{F}^a_{\mu\rho}\mathcal{F}^a_{\nu\sigma}, \tag{1}$$

where $R_{f(g)}$ is the curvature scalar of the strong (weak) gravity:

$$R_f = f^{\mu\nu}R^f_{\mu\nu}, \qquad R_g = g^{\mu\nu}R^g_{\mu\nu}; \tag{2}$$
$$\mathcal{F}^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e\epsilon^{abc}W^b_\mu W^c_\nu; \qquad a, b, c = 1, 2, 3. \tag{3}$$

Here $f^{\mu\nu}$ and $g^{\mu\nu}$ are the symmetric tensor metric fields, which describe, according to the model (Isham *et al* 1971), the strong and gravitational interactions, respectively; $k_f \sim m^{-1} \sim 10^{-14}$ cm is a typical distance of strong interactions, $k_g \sim 10^{-33}$ cm is a typical distance of gravitational interactions; $R^f_{\mu\nu}(R^g_{\mu\nu})$ is the Ricci tensor constructed by means of the $f(g)$ field. The f–g mixing (second term in (1)) leads to the emergence of mass of the f field (corresponding to the finite-range nature of strong interactions) whereas the g field remains massless (corresponding to the infinite-range nature of gravitational interactions).

According to the model (Isham *et al* 1971) strong (weak) gravity couples directly to the energy–momentum tensor of hadronic (leptonic) matter. Since we solve the problem in the lowest order of the quotient k_g/k_f, the effects of the weak gravity would be negligible. Thus, we can take the g field to be equal to Minkowski metric: $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and the next to the last term in (1) vanishes. The last term in (1) describes hadronic matter, which we have taken to consist of the Yang–Mills fields (gluons) without quarks.

We are interested in the regular static spherically symmetric solutions for the gauge field W^a_μ and the strong gravity metric field $f^{\mu\nu}$.

The most general and consistent way to write solutions for the f field we are looking for is as follows:

$$f^{00} = \mathrm{e}^{-u(r)}, \qquad f^{0i} = 0, \qquad f^{ij} = -\left(\mathrm{e}^{-v(r)}\delta^{ij} - (\mathrm{e}^{-v(r)} - \mathrm{e}^{-w(r)})\frac{x^i x^j}{r^2}\right) \tag{4}$$

where $i, j = 1, 2, 3$ and the vanishing of f^{0i} is due to the requirement of a static solution.

By the Wu–Yang *ansatz* we write W^a_μ as

$$W^a_i = \epsilon^{iaj}x^j(er^2)^{-1}p(r), \qquad W^a_0 = 0. \tag{5}$$

Using equations (1), (4) and (5), we obtain the energy of the system of static metric and gauge fields in terms of the independent scalar functions u, w, v and p:

$$E=-\int \mathscr{L}\,\mathrm{d}^3x=\frac{4\pi}{k_f^2}\int\Big[(\mathrm{e}^{(u+w)/2}-\mathrm{e}^{(u-w)/2}\,\mathrm{e}^{v})(u'+w')r-\mathrm{e}^{(u-w)/2}\,\mathrm{e}^{v}(\tfrac{1}{2}v'^2+v'u')r^2$$
$$+m^2\,\mathrm{e}^{(u+w)/2}\,\mathrm{e}^{v}(-\tfrac{1}{2}\mathrm{e}^{-u-w}-\tfrac{1}{2}\mathrm{e}^{-2v}-\mathrm{e}^{-u-v}-\mathrm{e}^{-w-v}+\tfrac{3}{2}\mathrm{e}^{-u}+\tfrac{3}{2}\mathrm{e}^{-w}$$
$$+3\mathrm{e}^{-v}-3)r^2+\frac{k_f^2}{e^2}\,\mathrm{e}^{(u+w)/2}\Big(\frac{1}{2r^2}\,\mathrm{e}^{-v}p^2(2+p)^2+\mathrm{e}^{-w}p'^2\Big)\Big]\,\mathrm{d}r. \tag{6}$$

Setting the variation of equation (6) equal to zero leads to the equations determining the functions u, w, v and p. Linearising these equations in u, w and v, we obtain:

$$-v''+\frac{w'}{r}-3\frac{v'}{r}+\frac{w}{r^2}-\frac{v}{r^2}+m^2(\tfrac{1}{2}w+v)=\frac{k_f^2}{2e^2r^4}[\tfrac{1}{2}p^2(2+p)^2+r^2p'^2], \tag{7}$$

$$-\frac{u'}{r}-\frac{v'}{r}+\frac{w}{r^2}-\frac{v}{r^2}+m^2(\tfrac{1}{2}u+v)=\frac{k_f^2}{2e^2r^4}[\tfrac{1}{2}p^2(2+p)^2-r^2p'^2], \tag{8}$$

$$-u''-v''-\frac{u'}{r}+\frac{w'}{r}-2\frac{v'}{r}+m^2(u+w+v)=-\frac{k_f^2}{2e^2r^4}p^2(2+p)^2, \tag{9}$$

$$r^2p''+\tfrac{1}{2}r^2p'(u'-w')=2p+3p^2+p^3. \tag{10}$$

Asymptotics at $r\to\infty$ of the linearised equations of the f–g gravity in the absence of matter (equations (7)–(9) with right-hand sides equal to zero) have been found (Aragone and Chela-Flores 1972) to approach the Minkowski metric exponentially. The presence of the gauge fields results (as we shall see) in the existence of solutions of the system (7)–(10) regular in the whole domain.

3. Solutions

Before going to the analysis of the solutions of (7)–(10), we shall make some observations about the solution of the equation for the pure Yang–Mills field in the flat space-time (equation (10) with $u=w=v=0$). This equation has three exact solutions: $p=0$, $p=-2$, and $p=-1$ (the first two with finite energies, the third with infinite energy). It is easy to see that the field strength $\mathscr{F}_{ij}^a$ vanishes, when $p=0, -2$. In other words, these two solutions are the vacuum solutions and therefore they can be obtained from one another using the gauge transformation:

$$\tilde{W}_j^a\frac{\sigma_a}{2}=U^{-1}W_j^a\frac{\sigma_a}{2}U+\frac{\mathrm{i}}{e}U^{-1}\partial_jU. \tag{11}$$

To see this, we choose the operator U in the form required by the symmetry of the problem:

$$U=\exp\Big(\mathrm{i}\beta\frac{x^k\sigma^k}{2r}\Big). \tag{12}$$

Setting $\tilde{W}_j^a=0$, we obtain from (11) the general form of the vacuum solution for the gauge field W_j^a:

$$W_j^a=-\frac{2\sin^2\frac{1}{2}\beta}{er^2}\epsilon^{jak}x^k+\frac{\sin\beta}{er}\Big(\delta^{ja}-\frac{x^jx^a}{r^2}\Big). \tag{13}$$

In the gauges $\beta=0$ and $\beta=\pi$ we have the solutions corresponding to $p=0$ and $p=-2$, respectively†. For definiteness we shall take the gauge $\beta=0$ which corresponds to the vacuum solutions $W_j^a=0$.

We have found that the regular solutions of the system of equations (7)–(10) in the chosen gauge have the following asymptotic behaviour at $r\to\infty$:

$$u=-3c^2(k_f/r)^6, \qquad w=-c^2(k_f/r)^6, \qquad v=2c^2(k_f/r)^6, \tag{14}$$

$$p=-ce(mk_f^2/r); \tag{15}$$

and at $r\to 0$:

$$u=\tilde{c}^2(r/k_f)^2, \qquad w=(1+3n)\tilde{c}^2(r/k_f)^2, \qquad v=n\tilde{c}^2(r/k_f)^2, \tag{16}$$

$$p=-\tilde{c}e(r/k_f)^2; \tag{17}$$

where c, $\tilde{c}$, n are the dimensionless constants of first order. The qualitative form of the solution for $p(r)$ is illustrated in figure 1.

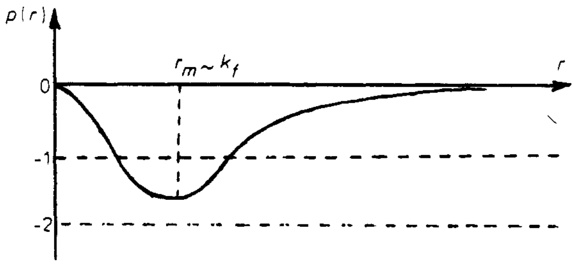

Figure 1. A qualitative form of the regular solution for $p(r)$. The behaviour of $p(r)$ in the non-asymptotic region is obtained using equation (10) with the assumption that u', $w' \ll 1$; the curve $p(r)$ has points of inflection at $p=-1$; oscillations of the curve around $p=-1$ are possible but unlikely.

Asymptotics of the solution at $r\to 0$ are determined by the terms in (7)–(10) independent of m, and, therefore, (16) and (17) remain unchanged in the limit $m\to 0$. However, asymptotics of the solutions at $r\to\infty$ differ for the cases of strong and weak (conventional) gravity‡. Instead of (14) we find the last case decreases like: $(k_g/r)^4$. Finally, we remark that this analysis can be extended for the cases of the gauge fields with higher symmetries (for example, SU(3)) which are more relevant to strong interactions.

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† Note in this connection that in the paper of Hsu (1976) the solution $p=-2$ is incorrectly interpreted as the monopole solution with zero mass.
‡ We recall also that in the case of weak (massless) gravity the metric field is parametrised by two independent scalar functions, i.e. we can put $v=0$.

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